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The SU(3) spin chain sigma model and string theory

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Abstract: The ferromagnetic integrable SU(3) spin chain provides the one loop anomalous dimension of single trace operators involving the three complex scalars of $\mathcal{N} = 4$ supersymmetric Yang-Mills. We construct the non-linear sigma model describing the continuum limit of the SU(3) spin chain. We find that this sigma model corresponds to a string moving with large angular momentum in the five-sphere in $AdS_5 \times S^5$. The energy and spectrum of fluctuations for rotating circular strings with angular momenta along three orthogonal directions of the five-sphere is reproduced as a particular case from the spin chain sigma model.

Keywords: String Duality, Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

A complete formulation of the AdS/CFT correspondence requires a precise identification of states on the string theory side with local gauge invariant operators on the dual field theory. But describing this identification in detail is truly involved because it implies both understanding the quantization of the string action in $AdS_5 \times S^5$, which remains a complicated problem, and obtaining the whole spectrum of $\mathcal{N} = 4$ supersymmetric Yang-Mills operators, which is difficult to compute. These difficulties were however overcome after the observation that there is a maximally supersymmetric plane-wave background for the IIB string [1], that can be readily quantized in the light-cone gauge [2]. As this plane-wave geometry is obtained through a limit of the $AdS_5 \times S^5$ background, the dual description in terms of a supersymmetric gauge theory must also involve some sort of equivalent limit. The plane-wave string/gauge theory duality is perturbatively reachable from both sides of the correspondence, and provides a precise identification relating string states to operators on the field theory side carrying a large charge [3].

These gauge theory operators are of the form $\mathrm{Tr}(X_1^J \ldots)$, with X_1 one of the $\mathcal{N} = 4$ complex scalars and $J \gg 1$, and where the dots stand for insertions of few other Yang-Mills fields. They have a dual realization as small closed strings whose center of mass is moving with large angular momentum J along some circle of S^5, and are localized at the center of AdS_5 [4]. Similarly, operators composed of the three $\mathcal{N} = 4$ complex scalars, $\mathrm{Tr}(X_1^{J_1} X_2^{J_2} X_3^{J_3})$, were proposed to correspond to semiclassical string solutions with three large angular momenta on the five-sphere, J_1, J_2 and J_3 [5].

In order to prove the duality a comparison must therefore be performed of the spectrum of string excitations with the anomalous dimensions of the corresponding field theory operators. However, once quantum corrections are taken into account there is operator mixing, and in order to construct generic operators and evaluate their anomalous dimension one should consider a one loop mixing for a large number of operators [6]. A brilliant

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insight into this tough obstacle came from the observation that the planar one loop anomalous dimension operator in the scalar sector of $\mathcal{N}=4$ supersymmetric Yang-Mills is the hamiltonian of an integrable SO(6) spin chain [7]. Anomalous dimensions of Yang-Mills operators carrying large charges can then be calculated by solving the thermodynamic Bethe ansatz, and successfully compared to the energy of string solitons [8, 9]. The integrability structure in $\mathcal{N}=4$ theory is not restricted to the scalar sector. Indeed, at one loop, it has been proved to extended to all conformal operators [10], leading to an integrable PSU(2,2|4) spin chain [11]. These observations have motivated a wide variety of related developments [12]–[21].

A natural question is then the relation of the integrable spin chain systems to the string non-linear sigma model. This problem was addressed in [22], where the continuum limit of the SU(2) Heisenberg spin chain was shown to reproduce the action describing strings rotating with large angular momentum in an S^3 section of S^5. This identification provides a very powerful tool for analyzing the integrable structures that arise on both sides of the correspondence, as well as for improving our understanding of the AdS/CFT correspondence itself. A recent paper [23] has shown that the agreement between the continuum limit of the spin chain and the string action holds to two loops in the SU(2) sector. Moreover, it has been proved that the thermodynamic limit of the Bethe equations in this sector precisely coincide with the classical results derived from the string sigma model up to two loops [24].

The SU(2) Heisenberg spin chain accounts for the anomalous dimensions of $\mathcal{N}=4$ single trace operators of the form $\mathrm{Tr}(X_1^{J_1}X_2^{J_2})$ and permutations. The aim of this paper is to extend the work of [22, 23] and study the continuum limit of the SU(3) spin chain describing the operators composed of arbitrary combinations of the three $\mathcal{N}=4$ complex scalars. Our analysis will consider only one loop contributions to the dilatation operator. The plan of the paper is the following. In section 2 we will obtain the non-linear sigma model associated to the continuum limit of the ferromagnetic SU(3) spin chain. In section 3 we will show that the sigma model derived from the spin chain reproduces the motion of a string with large angular momenta along three orthogonal directions of the five-sphere. A particularly interesting class of solitons in $AdS_5\times S^5$ is that of circular strings [5, 25]. In section 4 we will recover their energy and spectrum of fluctuations from the spin chain sigma model. We will end with some conclusions and further directions of research in section 5.

2. Ferromagnetic Heisenberg chain

The problem of finding the spectrum of anomalous dimensions for $\mathcal{N}=4$ Yang-Mills operators $\mathrm{Tr}(X_1^{J_1}X_2^{J_2}X_3^{J_3})$, or any of its permutations, can be mapped to that of solving an integrable spin chain where at each site sits a fundamental representation of SU(3). The correspondence between Yang-Mills operators and spin chain configurations is

$$\mathrm{Tr}\left(X_{i_1}X_{i_2}X_{i_3}X_{i_4}\ldots\right)\;\rightarrow\;|i_1\,i_2\,i_3\,i_4\ldots\rangle\,, \tag{2.1}$$

where $|i_l\rangle$ $(i_l=1,2,3)$ expand the fundamental representation of SU(3) at the l^{th} site.

The spin chain can be mapped to a discrete sigma model by introducing at each site a continuum set of variables. This is reviewed in detail in many textbooks for the simpler case of the SU(2) spin s chain [26]. An infinite set of spin states is obtained by applying an arbitrary SU(2) rotation to the maximally polarized state at each site. There is an $U(1)$ subgroup of rotations that just multiplies it by a phase, thus mapping this state to itself. Hence two angular variables are enough to label the resulting set of states. For $s = 1/2$, which is the relevant case for $\mathcal{N} = 4$ supersymmetric Yang-Mills, we have

$$|\hat{n}\rangle = \cos\psi\, e^{i\varphi}\,|1\rangle + \sin\psi\, e^{-i\varphi}\,|2\rangle\,, \tag{2.2}$$

where the states $|1\rangle$ and $|2\rangle$ expand the fundamental representation of SU(2). The central property that makes the overcomplete set $|\hat{n}\rangle$ relevant, is that they provide a resolution of the identity

$$\int d\mu(\hat{n})\;|\hat{n}\rangle\langle\hat{n}| = \mathbb{1}\,. \tag{2.3}$$

The appropriate measure over the continuum variables $\psi \in [0, \pi/2]$ and $\varphi \in [0, \pi]$ is

$$d\mu(\hat{n}) = \frac{4}{\pi}\sin\psi\cos\psi\; d\psi\; d\varphi\,. \tag{2.4}$$

A path-integral analysis of the partition function, with the help of (2.3), shows the equivalence between the spin chain and a discrete sigma model with action

$$S = \sum_{l=1}^{J}\int dt\;\left[\omega_l(t) - \mathcal{H}_l(t)\right], \tag{2.5}$$

with

$$\begin{aligned}\omega_l(t) &= \arg\frac{d}{d\rho}\,\langle\hat{n}_l(t)|\hat{n}_l(t+\rho)\rangle|_{\rho=0}\,,\\ \mathcal{H}_l(t) &= \langle\hat{n}_l(t),\hat{n}_{l+1}(t)|H_{l,l+1}|\hat{n}_l(t),\hat{n}_{l+1}(t)\rangle\,,\end{aligned} \tag{2.6}$$

for a chain governed by nearest neighbor interactions, $H = \sum H_{l,l+1}$. The continuum limit of this discrete sigma model arises in the thermodynamic limit of the chain, where the number of sites $J \to \infty$, and only slowly varying spin configurations are considered, allowing to replace finite differences between spin variables at neighboring sites by derivatives.

In this section we will construct the non-linear sigma model that describes the continuum limit of the SU(3) spin chain based on the fundamental representation. We define a set of spin coherent states at each site by applying a generic SU(3) rotation to an eigenstate of the Cartan generators. There is an SU(2) subgroup of rotations that leaves it invariant plus an additional $U(1)$ subgroup that just multiplies it by a phase. Thus we need four angular variables to label the continuum set of states

$$|\hat{n}\rangle = \cos\theta\,\cos\psi\, e^{i\varphi}\,|1\rangle + \cos\theta\,\sin\psi\, e^{-i\varphi}\,|2\rangle + \sin\theta\, e^{i\phi}\,|3\rangle\,. \tag{2.7}$$

The variables ϕ and φ are related to rotations generated by the two Cartan elements of SU(3), while θ and ψ correspond to rotations generated by two non-Cartan elements acting on different subspaces. The states $|\hat{n}\rangle$ form an overcomplete basis, with the resolution of identity

$$\int d\mu(\hat{n})\; |\hat{n}\rangle\langle\hat{n}| = \mathbb{1}\,, \tag{2.8}$$

where now the measure is

$$d\mu(\hat{n}) = \frac{12}{\pi^2}\sin\theta\cos^3\theta\sin\psi\cos\psi\, d\theta\, d\psi\, d\phi\, d\varphi\,. \tag{2.9}$$

Ranges of these variables are $\theta, \psi \in [0, \pi/2]$ and $\phi+\varphi, \phi-\varphi \in [0, 2\pi]$.

We analyze next the hamiltonian of the SU(3) spin chain. The hamiltonian of the SO(6) closed quantum spin chain with nearest neighbor interactions was recovered in [7] from the planar one loop anomalous dimension operator in the scalar sector of $\mathcal{N}=4$ Yang-Mills

$$H = \frac{\lambda}{16\pi^2}\sum_{l=1}^{J}(K_{l,l+1} + 2 - 2P_{l,l+1})\,. \tag{2.10}$$

The trace operator $K_{l,l+1}$ is zero when acting on the SU(2) or SU(3) sectors, namely on operators composed out of the $\mathcal{N}=4$ complex scalars without inclusion of their complex conjugates. Thus the hamiltonian in this sectors reduces to

$$H = \frac{\lambda}{8\pi^2}\sum_{l=1}^{J}(1 - P_{l,l+1})\,, \tag{2.11}$$

where $P_{l,l+1}$ is the permutation operator between the sites l and $l+1$. For the SU(2) case (2.11) is just the hamiltonian for the ferromagnetic XXX Heisenberg spin chain.

The symmetry group of the chain can be extended from SU(3) to $U(3)$ by adding a trivial $U(1)$. This $U(1)$ acts with the same rotation on each site of the chain. Its associated charge is the total number of sites, J. A basis of $U(3)$ generators acting on each site is

$$(T^{ab})_{ij} = \delta^a_i\,\delta^b_j\,, \tag{2.12}$$

where $a, b, i, j = 1, 2, 3$. In terms of this basis the permutation operator is given by

$$P_{l,l+1} = T_l^{ab}T_{l+1}^{ba}\,. \tag{2.13}$$

Let us consider two states, $|\alpha\rangle = \sum \alpha_i\,|i\rangle$ and $|\beta\rangle = \sum \beta_i\,|i\rangle$, located respectively at sites l and $l+1$ of the chain and compute the matrix element

$$\langle\alpha,\beta|P_{l,l+1}|\alpha,\beta\rangle = \langle\alpha|T_l^{ab}|\alpha\rangle\;\langle\beta|T_{l+1}^{ba}|\beta\rangle = |\langle\alpha|\beta\rangle|^2\,. \tag{2.14}$$

Therefore, in order to derive the sigma model associated to the chain, all what we need to analyze is the scalar product for two coherent states of the form (2.7)

$$\begin{aligned}\langle\hat{n}'|\hat{n}\rangle \;=\; &\sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta'\cos(\psi-\psi')\cos(\varphi-\varphi') + \\ &+ i\left[\sin\theta\sin\theta'\sin(\phi-\phi') + \cos\theta\cos\theta'\cos(\psi+\psi')\sin(\varphi-\varphi')\right]. \end{aligned}\tag{2.15}$$

From (2.5) and (2.6), in the long wavelength limit of the chain, where $|\hat{n}\rangle - |\hat{n}'\rangle = |\delta\hat{n}\rangle$, we obtain

$$
\begin{aligned}
S = &\frac{J}{2\pi}\int d\sigma\, dt \left(\sin^2\theta\dot{\phi} + \cos^2\theta\,\cos(2\psi)\dot{\varphi}\right) - \\
&-\frac{\lambda}{4\pi J}\int d\sigma\, dt\left[\theta'^2 + \cos^2\theta\,\left(\psi'^2 + \sin^2(2\psi)\,\varphi'^2\right) + \sin^2\theta\,\cos^2\theta\,\left(\phi' - \cos(2\psi)\,\varphi'\right)^2\right],
\end{aligned} \tag{2.16}
$$

where the dot and prime stand for derivatives with respect to t and σ, respectively. The result for the SU(2) sigma model [22] can be recovered by setting $\theta = 0$. The kinetic piece in the action is a Wess-Zumino term. The trace in the Yang-Mills operators implies that we should consider only periodic spin configurations invariant under the shift operator [7]. Namely, we require $|\bar{n}(2\pi, t)\rangle = |\bar{n}(0, t)\rangle$, where these states are defined by the choice of global phase $|\bar{n}\rangle = e^{-i\phi}|\hat{n}\rangle$, together with [24, 22]

$$
\int d\sigma\, \cos^2\theta(\phi' - \cos(2\psi)\varphi') = 2\pi s\,, \tag{2.17}
$$

with s integer.

The action (2.16) is invariant under constant shifts in ϕ and φ. The corresponding conserved angular momenta are

$$
P_\phi = \frac{J}{2\pi}\int d\sigma \sin^2\theta\,, \qquad P_\varphi = \frac{J}{2\pi}\int d\sigma\cos^2\theta\cos(2\psi)\,, \tag{2.18}
$$

and the hamiltonian is

$$
H = \frac{\lambda}{4\pi J}\int d\sigma\,\left[\theta'^2 + \cos^2\theta\,\left(\psi'^2 + \sin^2(2\psi)\,\varphi'^2\right) + \frac{1}{4}\sin^2(2\theta)\,\left(\phi' - \cos(2\psi)\,\varphi'\right)^2\right]. \tag{2.19}
$$

3. String sigma model

We will now describe how the non-linear sigma model for a string rotating in S^5 becomes the ferromagnetic sigma model for the SU(3) spin chain, after some adequate large angular momentum limit is taken. Let us then consider the propagation of the dual string on S^5. The associated metric, including the decoupled time coordinate t,

$$
ds^2 = -dt^2 + d\theta^2 + \sin^2\theta\, d\phi_3^2 + \cos^2\theta\left(d\psi^2 + \cos^2\psi\, d\phi_1^2 + \sin^2\psi\, d\phi_2^2\right), \tag{3.1}
$$

can be more conveniently written if we define

$$
\phi_1 = \alpha + \varphi\,, \qquad \phi_2 = \alpha - \varphi\,, \qquad \phi_3 = \alpha + \phi\,. \tag{3.2}
$$

The metric then becomes

$$
\begin{aligned}
ds^2 \;=\; &-dt^2 + d\theta^2 + \sin^2\theta\, d\phi^2 + 2\sin^2\theta\, d\phi\, d\alpha + d\alpha^2 + \\
&+ \cos^2\theta\left(d\psi^2 + 2\cos(2\psi)\, d\varphi\, d\alpha + d\varphi^2\right).
\end{aligned} \tag{3.3}
$$

Introducing one more change of variables, $\alpha \to \alpha + t$, we obtain

$$
\begin{aligned}
ds^2 \;=\; &d\theta^2 + \sin^2\theta\, d\phi^2 + 2\sin^2\theta\, d\phi\,(dt + d\alpha) + 2dt\, d\alpha + d\alpha^2 + \\
&+ \cos^2\theta\left(d\psi^2 + 2\cos(2\psi)\, d\varphi\,(dt + d\alpha) + d\varphi^2\right).
\end{aligned} \tag{3.4}
$$

We compute next the Polyakov action in this background. We will choose coordinates so that $t = \kappa\tau$. An interesting limit to consider is $\kappa \to \infty$, while $\kappa\,\partial_\tau X^\mu$ (with $X^\mu \neq t$) is kept fixed [22]. This limit corresponds to a string whose motion is mainly absorbed by the shift in the coordinate α, and gives rise to large spins: $\partial_\tau \phi_i = \kappa + \mathcal{O}(1/\kappa)$. In this limit the action describing a string in the background (3.4) simplifies to

$$S = \frac{R^2}{4\pi\alpha'} \int d\sigma d\tau [2\kappa\partial_\tau\alpha + 2\kappa\cos^2\theta\cos(2\psi)\,\partial_\tau\varphi + 2\kappa\sin^2\theta\,\partial_\tau\phi - \alpha'^2 - \theta'^2 - \\ - \cos^2\theta\varphi'^2 - \sin^2\theta\phi'^2 - \cos^2\theta\psi'^2 - 2\sin^2\theta\alpha'\phi' - 2\cos^2\theta\cos(2\psi)\,\alpha'\varphi'] \,, \tag{3.5}$$

and the Virasoro constraints, $G_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X^\nu = 0$ and $G_{\mu\nu}\partial_\tau X^\mu \partial_\tau X^\nu + G_{\mu\nu}\partial_\sigma X^\mu \partial_\sigma X^\nu = 0$, reduce, respectively, to

$$2\kappa\alpha' + 2\kappa\cos^2\theta\cos(2\psi)\varphi' + 2\kappa\sin^2\theta\,\phi' = 0\,, \tag{3.6}$$

and

$$2\kappa\,\partial_\tau\alpha + 2\kappa\cos^2\theta\cos(2\psi)\,\partial_\tau\varphi + 2\kappa\sin^2\theta\,\partial_\tau\phi + \alpha'^2 + \theta'^2 + \cos^2\theta\,\varphi'^2 + \\ + \sin^2\theta\,\phi'^2 + \cos^2\theta\,\psi'^2 + 2\sin^2\theta\,\alpha'\phi' + 2\cos^2\theta\cos(2\psi)\,\alpha'\varphi' = 0\,. \tag{3.7}$$

If we insert (3.6) in (3.5), and change variables to $t = \kappa\tau$, we get

$$S = \frac{R^2\kappa}{2\pi\alpha'} \int d\sigma\,dt \left[\dot\alpha + \cos^2\theta\cos(2\psi)\,\dot\varphi + \sin^2\theta\dot\phi\right] - \\ - \frac{R^2}{4\pi\alpha'\kappa} \int d\sigma\,dt\left[\theta'^2 + \cos^2\theta(\psi'^2 + \sin^2(2\psi)\,\varphi'^2) + \sin^2\theta\cos^2\theta(\phi' - \cos(2\psi)\,\varphi')^2\right]. \tag{3.8}$$

The momentum associated to α is $P_\alpha = \kappa R^2/\alpha'$. According to the change of coordinates (3.2), P_α should be identified with the total angular momentum, J. Therefore this action coincides precisely with the SU(3) spin chain sigma model action (2.16), once we use the gravity/gauge theory relation $R^2 = \alpha'\sqrt{\lambda}$. Notice that a trivial variable playing the same role as α could have been introduced also in the spin chain sigma model, where it would correspond to a global phase multiplying the state $|\hat n\rangle$. The measure on the overcomplete set $\{|\hat n\rangle\}$ introduced in the previous section (2.9) is just (proportional to) the volume element of the five-sphere.

4. Circular strings

In this section we will concentrate on a particular family of classical string solutions, that of circular strings rotating on S^5 and located at the center of AdS_5 [5, 25]. Using the spin chain sigma model we will reobtain their energy and spectrum of fluctuations at first order in the effective coupling constant $\frac{\lambda}{J^2}$, verifying that the limit we have taken in the string sigma model correctly reproduces the known results for large angular momentum. It would be interesting to extend this analysis to other classes of solutions (see [27] for a complete account of semiclassical string solutions).

The equations of motion for the spin chain sigma model (2.16) are

$$\begin{aligned}\sin(2\psi)\left[\cos^2\theta\,\dot\psi+\frac{\lambda}{4J^2}\sin^2(2\theta)(\phi'-\cos(2\psi)\varphi')\psi'\right]+\frac{\lambda}{2J^2}\partial_\sigma\left[\cos^2\theta\sin^2(2\psi)\varphi'\right]&=0\,,\\ \sin(2\theta)\,\dot\theta-\frac{\lambda}{4J^2}\partial_\sigma\left[\sin^2(2\theta)(\phi'-\cos(2\psi)\varphi')\right]&=0\,,\\ \cos^2\theta\sin(2\psi)\left(\dot\varphi+\frac{\lambda}{J^2}\left[\cos^2\theta\,\cos(2\psi)\varphi'^2+\sin^2\theta\,\varphi'\phi'\right)\right]\right)-\frac{\lambda}{2J^2}\partial_\sigma\left[\cos^2\theta\psi'\right]&=0\,,\\ \frac{\lambda}{J^2}\theta''+\sin(2\theta)\left(\dot\phi-\cos(2\psi)\dot\varphi+\frac{\lambda}{2J^2}\left[\psi'^2+\sin^2(2\psi)\varphi'^2-\cos(2\theta)\left(\phi'-\cos(2\psi)\varphi'\right)^2\right]\right)&=0\,.\end{aligned} \tag{4.1}$$

Circular string solutions correspond to the ansatz $\theta=\theta_0$ and $\psi=\psi_0$ constant, and $\partial_\sigma\varphi=m$, $\partial_\sigma\phi=n$, where m and n are both integers or half-integers fulfilling (2.17). This ansatz solves the previous equations with

$$\begin{aligned}\dot\varphi&=-\frac{\lambda}{J^2}m\left[m\left(\frac{J_1}{J}-\frac{J_2}{J}\right)+n\frac{J_3}{J}\right],\\ \dot\phi&=\frac{\lambda}{2J^2}\left(n^2-m^2-2n\left[m\left(\frac{J_1}{J}-\frac{J_2}{J}\right)+n\,\frac{J_3}{J}\right]\right),\end{aligned} \tag{4.2}$$

where we have made use of the relations

$$\frac{J_1-J_2}{J_1+J_2}=\cos(2\psi_0)\,,\qquad \frac{J_3}{J}=\sin^2\theta_0\,, \tag{4.3}$$

derived from the conserved momenta (2.18), $P_\varphi=J_1-J_2$ and $P_\phi=J_3$. We obtain the following expression for the energy

$$E=\frac{\lambda}{2J}\frac{1}{J^2}\left[(2m)^2J_1J_2+(n-m)^2J_1J_3+(n+m)^2J_2J_3\right]. \tag{4.4}$$

Equations (4.2) and (4.4) reproduce the angular velocities and energy of a circular string rotating along three orthogonal directions of S^5 [25], provided we identify

$$2m=m_1-m_2\,,\qquad n-m=m_3-m_1\,,\qquad n+m=m_3-m_2\,. \tag{4.5}$$

If we further equal $s=m_3$, (2.17) reproduces the well-known constraint of circular strings $\sum_{i=1}^3 m_iJ_i=0$. Notice that (4.5) is consistent with (3.2); an analogous relation applies for the angular velocities.

The study of the spectrum of quadratic fluctuations around these solutions allows to determine their stability and to derive the one loop sigma model corrections to the energy [28, 25]. In the limit of large angular momenta the spectrum of excitations contains modes with low frequencies $\omega\sim\frac{\lambda}{J^2}$, and modes with frequencies $\omega\sim 1$. We will now check that the low frequency fluctuations can also be derived from the spin chain sigma model.

Let us first consider a two-spin circular string. They are obtained by setting in the previous ansatz $\theta_0=0$, which implies $J_3=0$. With this restriction, equations (4.1) reduce to those of the SU(2) spin chain sigma model [22]. Quadratic fluctuations around circular two-spin solutions are governed by the equations

$$\begin{aligned} \dot{\psi} + \frac{\lambda}{2J^2}\left(4m\cos(2\psi_0)\,\psi' + \sin(2\psi_0)\,\varphi''\right) &= 0\,, \\ \sin(2\psi_0)\dot{\varphi} - \frac{\lambda}{2J^2}\left(\psi'' + 4m^2\sin^2(2\psi_0)\,\psi - 2m\sin(4\psi_0)\,\varphi'\right) &= 0\,. \end{aligned} \tag{4.6}$$

We easily obtain the spectrum

$$\omega = \frac{\lambda}{2J^2} r\left[-4m\cos(2\psi_0) \pm \sqrt{r^2 - 4m^2\sin^2(2\psi_0)}\right], \tag{4.7}$$

with r integer, which reproduces the results in [25].

The analysis of fluctuations around generic three-spin circular solutions is straightforward but tedious. Here we will only consider the simplified case when $J_1 = J_2$ and $n = 0$. In terms of the string sigma model variables this translates into $J_1 = J_2$ and $m_3 = 0$, which was analyzed in [28]. The linearized equations of motion around this solution are

$$\begin{aligned} \frac{\lambda}{J^2}\,\theta'' + \sin(2\theta_0)\,\dot{\phi} + \frac{\lambda}{J^2}\,m\sin(2\theta_0)\,\varphi' &= 0\,, \\ \dot{\theta} - \frac{\lambda}{4J^2}\sin(2\theta_0)\,\left(\phi'' + 2m\psi'\right) &= 0\,, \\ \dot{\psi} - \frac{\lambda}{2J^2}\,\left(2m\tan\theta_0\,\theta' - \varphi''\right) &= 0\,, \\ \dot{\varphi} - \frac{\lambda}{J^2}\left(2m^2\cos^2\theta_0\,\psi - m\sin^2\theta_0\,\phi'\right) - \frac{\lambda}{2J^2}\,\psi'' &= 0\,. \end{aligned} \tag{4.8}$$

We obtain the following spectrum of fluctuations

$$\omega^2 = \left(\frac{\lambda}{2J^2}\right)^2 r^2\left[r^2 + 2m^2\left(3\sin^2\theta_0 - 1\right) \pm 2m\sqrt{\left(3\sin^2\theta_0 - 1\right)^2 m^2 + 4\sin^2\theta_0 (r^2 - m^2)}\right], \tag{4.9}$$

in agreement with [28].

The stability properties of the circular strings are determined by the low frequency fluctuations. This is not the case for the one loop corrections to their energy. Although $\Delta E/E \sim 1/J$, the contributions to ΔE are not saturated by the modes we have obtained above [28, 25]. It would be interesting to see whether the one loop correction to the classical energy of circular strings can be also derived in the framework of the spin chain sigma model.

5. Conclusions

The motivation of this note has been to explore the relation of the integrable SU(3) spin chain with the string non-linear sigma model. We have in particular precisely identified the ferromagnetic sigma model describing the continuum limit of the SU(3) chain with the sigma model for a string in S^5. In order to understand and clarify this equivalence we have also seen how classical solutions to the spin chain sigma model correspond to string configurations in the dual $AdS_5 \times S^5$ background.

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In a recent paper [23], it has been shown that the identification between the continuum limit of the spin chain and the string action extends to two loops in the SU(2) sector. The derivation of the spin chain sigma model, which to leading order requires considering only long wavelength spin configurations, beyond one loop involves quantum corrections from short wavelength configurations. Again for the SU(2) sector, the direct correspondence between solutions of thermodynamic Bethe ansatz equations and classical solutions of the string sigma model to two loop order has been proved in [24], and shows the equivalence between the integrable structures arising in the gauge and string theory sides (see also [16, 20, 29]).

It would be extremely interesting to extend the results in [23, 24] to the SU(3) sector. However, there is an important difference between the SU(2) and SU(3) scalar sectors of $\mathcal{N} = 4$ Yang-Mills. SU(2) operators mix only among themselves at all orders in perturbation theory, maintaining the notion of the length of the chain as a well defined concept. However, the SU(3) scalar sector is only closed at one loop. At higher orders transitions are possible between the configurations $X_1X_2X_3$ and $\Psi_1\Psi_2$, where Ψ_i are complex combinations of the four $\mathcal{N} = 4$ spinors. In order to obtain a closed sector, SU(3) has to be extended to SU(2|3), whose fundamental representation can be associated with both X_i and Ψ_j fields [10]. The spin chain corresponding to this sector has the striking property that the number of sites becomes a dynamical variable. In spite of that, arguments were presented in [30] in favor of its integrability. The study of this dynamical spin chain is important for a better understanding of the gauge theory/gravity correspondence.

One should at any rate stress that there is still no clear proof that integrability is valid in the planar limit of $\mathcal{N} = 4$ Yang-Mills beyond two loops [8, 31, 24, 29, 23], although we know that classical integrability of the string sigma model in $AdS_5 \times S^5$ holds to all orders [32]–[34]. The analysis of the continuum limit of the spin chain provides a very promising tool for addressing this problem, whose expected outcome is that the sigma model resulting from the spin chain including all higher loop corrections is the classical action of the string in $AdS_5 \times S^5$ [22, 23].

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